A note on independent set approximation

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1 November 2026

Context. Boppana and Halldórsson [1] gave a polynomial-time $O(n/\log^2 n)$-approximation for maximum independent set; Feige [2] improved this to $O(n(\log\log n)^2/\log^3 n)$, matching the earlier graph-coloring bound of Halldórsson [3]. Lovett [4] and then Song [5] obtain stretched-exponential savings in quasi-polynomial time. Their density-sensitive guarantees also imply polynomial-time improvements by a simple partition argument. I claim no original idea in this note: it records direct consequences of their results using elementary reductions.

All logarithms are base two. Approximation ratios are at least one: an R-approximation returns an independent set of size at least $\alpha(G)/R$, or a coloring with at most $R\chi(G)$ colors.

Theorem 1 (Consequence of Song)**.** *Assuming Song's Theorem 4.6 [5], maximum independent set and minimum graph coloring on n-vertex graphs have deterministic polynomial-time approximation algorithms with ratio*

$$\boxed{\frac{n}{2^{\Omega((\log n)^{1/3})}}.}$$

Proof. Set $t = \lfloor (\log n/288)^{1/3} \rfloor$, $s = 2^t$ and $m = 2^{144t^2}$, and partition the vertices into blocks of sizes in $[m, 2m]$: if $\alpha(G) \geq n/s$, the pigeonhole principle gives a block with independent-set density at least $1/s$, so Song's algorithm finds s independent vertices in total time $n^{O(1)}m^{O(t)} = n^{O(1)}$. Return the largest verified set, or a singleton; the cases $\alpha(G) \geq n/s$ and $\alpha(G) < n/s$ both give ratio at most n/s.

For coloring, repeatedly apply extraction to any m remaining vertices and give s returned vertices a fresh color; on failure or with fewer than m vertices remaining, use individual colors. If $\chi(G) \leq s$, every invocation succeeds and at most $n/s + m \leq 2n/s$ colors are used; if $\chi(G) > s$, even n colors give ratio below n/s. Calls are time-limited and outputs checked; finitely many inputs with $t = 0$ are handled separately. □

The scale of the improvement. Zuckerman [6] gives NP-hardness of $n^{1-\varepsilon}$-approximation for every fixed $\varepsilon > 0$. Under the stronger assumption that NP has no bounded-error randomized quasi-polynomial-time algorithms, Khot–Ponnuswami [7] rule out polynomial-time ratios

$$\frac{n}{2^{(\log n)^{3/4+\varepsilon}}}, \qquad 0 < \varepsilon < 1/4.$$

The polynomial-time upper bound is therefore now in the same qualitative ballpark as this hardness bound: both have a stretched-exponential saving over n, rather than a polylogarithmic saving. The exponents $1/3$ and $3/4 + \varepsilon$ still leave a substantial gap; this is not a matching bound. Lovett's theorem alone gives the analogous polynomial-time exponent $1/4$.

Why Song improves the exponent. Lovett uses a fixed marginal gain of $3/2$ per conditioning step and obtains a sparse set of size $N2^{-O(t^2h)}$ and density at most 2^{-h}. Song groups positive correlations by dyadic scale and charges logarithmic vertex loss to the actual logarithmic marginal gain, reducing the size loss to $2^{-O(th)}$. With $h = t+1$, blocks of size $2^{O(t^2)}$ suffice instead of $2^{O(t^3)}$; their running time is $2^{O(t^3)}$, which permits $t = \Theta((\log n)^{1/3})$ in polynomial time.

Density-dependent consequences. Song's stronger sparse-set lemma also improves the parameterized guarantees. Put $t = \lceil \log k \rceil$ for $2 \leq k \leq n$. Under $\alpha(G) \geq n/k$, choosing $h = \Theta(\log n/t)$ in that lemma and applying greedy extraction gives

$$|I| \geq \Omega\Big(n^{c_1/\log k}\Big) \quad \text{in time } n^{O(\log k)},$$

for an absolute $c_1 > 0$; integer rounding and the small-input case are absorbed in the constant. Applying the same argument to blocks of size about $n^{1/(2t)}$ yields the following uniform polynomial-time statement.

Corollary 2 (Consequence of Song's Lemma 4.5)**.** *There are absolute constants $c_0, c > 0$ and a deterministic algorithm, polynomial in n with degree independent of k, which, given $\alpha(G) \geq n/k$, finds an independent set of size at least $c_0 n^{c/(\log k)^2}$.*

The density promise passes to some block by averaging; the block algorithm costs $(n^{1/(2t)})^{O(t)} = n^{O(1)}$, and its output has the stated size. If the parameter choice makes $h < 1$, a singleton supplies the bound after adjusting c_0.

For example, the polynomial-time output guarantees are

Density parameter k	Independent-set size
fixed $k \geq 2$	$\Omega(n^{c/(\log k)^2})$
$(\log n)^2$	$2^{\Omega(\log n/(\log\log n)^2)}$
$2^{(\log n)^{1/3}}$	$2^{\Omega((\log n)^{1/3})}$

For $k = \log^2 n$, the unpartitioned algorithm instead gives $2^{\Omega(\log n/\log\log n)}$ vertices in time $n^{O(\log\log n)}$. These are output-size guarantees under a density promise. For comparison, Alon–Kahale [8] obtain polynomial-size independent sets using the basic theta SDP when the independence number exceeds n/k by a linear amount, for fixed k; Feige [2] obtains $\Omega((\log n/\log\log n)^2)$ vertices under an $n/\operatorname{polylog} n$ promise.

Personal note. Approximating independent set was the subject of my first result, with Ravi Boppana [1]. In my thesis [9, Conjecture 1, p. 51], I conjectured an $n/\operatorname{polylog} n$ approximation barrier for coloring and explicitly extended the conjecture to independent set, motivated by the paucity of effective combinatorial approaches. Yet I was certain that the eventual solution would involve semidefinite programming, which was not very developed at the time. These results overturn my conjectured scale while vindicating that intuition about the method.

References

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